Multivariate Krawtchouk polynomials and a spectral theorem for symmetric tensor powers

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Multivariate Krawtchouk polynomials are constructed.

A spectral theorem for associated quantum observables is presented.

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1 Krawtchouk polynomials in one variable and the binomial distribution

Krawtchouk polynomials may be defined via the generating function

$$(1+\lambda pv)^{N-j}(1-\lambda qv)^{j} = \sum_{0\le n\le N} v^n\, k_n(j,N)$$

The polynomials $k_n(j,N)$ are orthogonal with respect to the binomial distribution with parameters N, p.

➧ They are part of the legacy of Mikhail Kravchuk

N. Virchenko, et al., eds.
Development of the Mathematical Ideas of Mykhailo Kravchuk (Krawtchouk),
Shevchenko Scientific Society, Kyiv-New York, 2004.

Krawtchouk polynomials appear in diverse areas of mathematics and science. Applications range from coding theory to image processing.

Work with René Schott

- N. Botros, J. Yang, P. Feinsilver, and R. Schott, *Hardware Realization of Krawtchouk Transform using VHDL Modeling and FPGAs*, IEEE Transactions on Industrial Electronics, **49** 6 (2002)1306–1312.

- Ph. Feinsilver and R. Schott, *Finite-Dimensional Calculus*, Journal of Physics A: Math.Theor., **42**:375214, 2009.

- Ph. Feinsilver and R. Schott, *On Krawtchouk Transforms*, Intelligent Computer Mathematics, proceedings 10th Intl. Conf. AISC 2010, LNAI 6167, 64–75.

- F&S, *Algebraic Structures and Operator Calculus*, 3 vols., 1993-1995.

2 Symmetric tensor powers

Given a $d \times d$ matrix A, the action on the symmetric tensor algebra of the underlying vector space defines its second quantization or “symmetric representation”.

Introduce commuting variables $x_1, \ldots, x_d$. Map

$$y_i = \sum_j A_{ij} x_j$$

We will use multi-indices, $m = (m_1, \ldots, m_d)$, $m_i \geq 0$, similarly for n.

The induced map at level N has matrix elements $\bar{A}_{nm}$ determined by the expansion

$$y^n = y_1^{n_1} \cdots y_d^{n_d} = \sum_m \bar{A}_{nm} x^m \ .$$

The matrix $\bar{A}$ is often called the *induced matrix* at level N. The induced matrix maps monomials of homogeneous degree N to polynomials of homogeneous degree N.

2.1 Transpose Lemma

We introduce the special matrix B which is a diagonal matrix with multinomial coefficients as entries

$$B_{nm} = \delta_{nm} \binom{N}{n} = \frac{N!}{n_1!\, n_2! \, \cdots n_d!} \; .$$

B is the diagonal of the induced matrix at level N of the matrix consisting of all 1's.
The level N is implicit according to context.

If p is a diagonal matrix with entries $p_i > 0$, $\sum_i p_i = 1$, then the matrix

$$B\overline{p}$$

yields the probabilities for the corresponding multinomial distribution.

The map $A \to \bar{A}$ is at each level a multiplicative homomorphism,

$$\overline{A_1 A_2} = \bar{A}_1 \, \bar{A}_2 \; .$$

The main lemma is the relation between the induced matrix of A with that of its transpose, $A^\top$.

Transpose Lemma.

The induced matrices at each level satisfy

$$\overline{A^\top} = B^{-1} \bar{A}^\top B \; .$$

3 Construction of Krawtchouk polynomial systems

Start with U, an orthogonal (unitary) matrix.

Make all entries of first column positive by taking out phases and form the probability matrix thus

$$p = \begin{pmatrix} U_{00}^2 & & \\ & \ddots & \\ & & U_{d0}^2 \end{pmatrix} = \begin{pmatrix} p_0 & & \\ & \ddots & \\ & & p_d \end{pmatrix}$$

row and column indices running from 0 to d.

Define

$$A = \frac{1}{\sqrt{p}}\, U \sqrt{D}$$

where D is diagonal with all positive entries on the diagonal. The **essential property** satisfied by A is

$$A^\top p A = D\ .$$

3.1 Krawtchouk systems

In any degree N, the induced matrix $\bar{A}$ satisfies

$$\overline{A^\top}\bar{p}\bar{A} = \bar{D}\ .$$

Using the Transpose Lemma

$$B\overline{A^\top} = \bar{A}^\top B$$

with B the special multinomial diagonal matrix yields

$$\Phi\, B\bar{p}\,\Phi^\top = B\bar{D}$$

the **Krawtchouk matrix** Φ being thus defined as $\bar{A}^\top$.

The matrix elements, i. e. entries, of Φ are the values of the multivariate Krawtchouk polynomials thus determined.

The matrix $B\bar{D}$ is the diagonal matrix of squared norms according to the orthogonality of the Krawtchouk polynomial system.

4 Columns Theorem for symmetric powers

- **MacMahon's Master Theorem** yields the diagonal matrix elements of the symmetric tensor powers. Namely,

Let $U = \text{diag}(u_1, \ldots, u_d)$. Then, the coefficient of $u^m = u_1^{m_1} \cdots u_d^{m_d}$ in the expansion of $\det(I - UA)^{-1}$ is the diagonal matrix element $\bar{A}_{mm}$.

- **We present** a variation that reproduces all of the matrix elements.

Given a matrix A, with each column of A form a diagonal matrix. Thus,

$$\Lambda_j = \text{diag}\left((A_{ij})\right)$$

where

$$(\Lambda_j)_{ii} = A_{ij} \ .$$

➨ **Columns Theorem.**

For any matrix A, let Λ_j be the diagonal matrix formed from column j of A. Let

$$\Lambda = \sum v_j \Lambda_j \ .$$

Then the coefficient of v^n in the level N induced matrix $\bar{\Lambda}$ is a diagonal matrix with entries the n^{th} column of $\bar{A}$.

Proof: Setting $\vec{y} = \Lambda \vec{x}$, we have

$$y_k = (\sum v_j A_{kj}) x_k \ \Rightarrow \ y^m = (\sum \bar{A}_{mn} v^n)\, x^m \ .$$

A careful reading of the coefficients yields the result. $\square$

We may express this in the following useful way:

the diagonal entries of $\bar{\Lambda}$ are generating functions for the matrix elements of $\bar{A}$.

4.1 Quantum observables

Define observables by

$$X_j = A^{-1} \Lambda_j A \,.$$

Let $X = \sum v_j X_j$. Then

$$AX = \Lambda A$$

and the symmetric tensor powers satisfy

$$\bar{A}\bar{X} = \bar{\Lambda}\bar{A}$$

the induced spectral formula for $\bar{X}$.

4.2 Quantum random walks

• Write, the superscript denoting the level N symmetric tensor power,

$$(I + t \sum X_j)^{(N)} = \sum t^m \xi_m(N) .$$

So $\xi_m(N)$ is the sum of all elementary symmetric tensors of order N having exactly m factors not equal to the identity.

• For example, with a single $X_j = X$,

$$\xi_1(3) = X \otimes I \otimes I + I \otimes X \otimes I + I \otimes I \otimes X$$

the quantum random walk after three steps.

• Taking $\Lambda_0 = I$, $v_0 = 1$, and t for the remaining v_j's in the discussion above yields the spectral representation for the quantum random walks and their extension to higher levels.

4.3 Spectral representation as a recurrence formula

Now take A corresponding to a Krawtchouk system, with $\Phi = \bar{A}^\top$. Then

$$\bar{X}^\top \Phi = \Phi \bar{\Lambda}$$

with $\bar{X}^\top$ combining rows of Φ resulting in multiplying the entries of a given row according to the spectrum.

For $n = 1$, this is a recurrence formula for the corresponding orthogonal polynomials. Namely, it shows the effect of multiplying ϕ_m, say, by ϕ_1.

The higher powers of v yield higher-level recursion formulas. They correspond to *linearization formulas* of the type

$$\phi_n \phi_m = \sum_{\ell} c^{\ell}_{mn} \phi_\ell \ .$$

5 Contexts

Gaussian quadrature Let $\{\phi_0, \ldots, \phi_n\}$ be an orthogonal polynomial sequence with positive weight function on an interval I of the real line. For Gaussian quadrature,

$$\int_{\mathrm{I}} f \approx \sum_k w_k f(x_k)$$

with x_k the zeros of ϕ_n and appropriate weights w_k. Let

$$A_{ij} = \phi_{i-1}(x_j)$$

Then, with Γ the diagonal matrix of squared norms, $\Gamma_{ii} = \|\phi_i\|^2$, we have

$$AWA^\top = \Gamma$$

where W is the diagonal matrix with $W_{kk} = w_k$.

➧ **Association schemes** Given an association scheme with adjacency matrices A_i, the P and Q matrices correspond to the decomposition of the algebra generated by the A_i into an orthogonal direct sum, the entries P_{ij} being the corresponding eigenvalues. A basic result is the relation

$$P^\top D_\mu P = v\, D_v$$

where D_μ is the diagonal matrix of multiplicities and D_v the diagonal matrix of valencies of the scheme.

Work of Delsarte, Bannai, ...

♫ `Example`

Start with the orthogonal matrix

$$U = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} .$$

Factoring out the squares from the first column we have

$$p = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

The binomial coefficient matrix is $B = \text{diag}(1, 4, 6, 4, 1)$.

We have the Kravchuk matrix

$$\Phi = (A^{(4)})^\top = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 4 & 2 & 0 & -2 & -4 \\ 6 & 0 & -2 & 0 & 6 \\ 4 & -2 & 0 & 2 & -4 \\ 1 & -1 & 1 & -1 & 1 \end{pmatrix} .$$

The entries of p become $\bar{p} = \frac{1}{16} I_5$.

Take the second column of A and form the diagonal matrix

$$\Lambda_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

The corresponding observable is

$$X_1 = A^{-1}\Lambda_1 A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Let $\Lambda = I + v\Lambda_1$ and $X = I + v\,X_1$.

Then $\Lambda^{(4)} = \text{diag}\big($

$(1{+}v)^4, (1{+}v)^3(1{-}v), (1{+}v)^2(1{-}v)^2, (1{+}v)(1{-}v)^3, (1{-}v)^4\big)$

And $X^{(4)} =$

$$\begin{pmatrix} 1 & 4v & 6v^2 & 4v^3 & v^4 \\ v & 1+3v^2 & 3v+3v^3 & 3v^2+v^4 & v^3 \\ v^2 & 2v+2v^3 & 1+4v^2+v^4 & 2v+2v^3 & v^2 \\ v^3 & 3v^2+v^4 & 3v+3v^3 & 1+3v^2 & v \\ v^4 & 4v^3 & 6v^2 & 4v & 1 \end{pmatrix}$$

Now we have the spectrum via the coefficient of v in $\Lambda^{(4)}$

$$\text{Spec} = \text{diag}(4, 2, 0, -2, -4)$$

and the coefficient of t in the transpose of $X^{(4)}$ give the recurrence coefficients

$$\text{Rec} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 \\ 0 & 3 & 0 & 3 & 0 \\ 0 & 0 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

satisfying the relation

$$(\text{Rec})\,\Phi = \Phi\,(\text{Spec})$$

which is essentially the recurrence relation for the corresponding Krawtchouk polynomials.

6 Further aspects

- **We acknowledge** the seminal paper of R. C. Griffiths Orthogonal polynomials and multinomial distributions, Australian J. Stat. **13**(1971) 27–35.

- **As Bernoulli systems**, systems of orthogonal polynomials related to representations of the Heisenberg algebra, $\mathfrak{sl}(n)$, etc., with probabilistic interpretations relating to exponential martingales of associated processes.