Completely Simple Semigroups
Lie Algebras
and the Road Coloring Problem

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Consider a semigroup generated by matrices associated with an edge-coloring of a strongly connected, aperiodic digraph. We call the semigroup **Lie-solvable** if the Lie algebra generated by its elements is solvable. We show that if the semigroup is Lie-solvable then its kernel is a right group. Next, we analyze the Lie algebras generated by the kernel. The Lie structure of a subalgebra generated by two idempotents is completely described. Finally, we discuss an infinite class of examples that are shown to always produce strongly connected aperiodic digraphs.

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1 Semigroups and kernels

Coloring: $G = (V, \mathcal{E})$ is a digraph of uniform outdegree d. Any labeling of the edges with members of A, where $|A| = d$, such that each edge issuing from any given vertex has a distinct label is a *coloring* of the digraph.

Any such coloring uniquely determines an automaton $\delta : V \times A \to V$, where $R_a(v) = va$ is the terminal point of the directed edge with initial point v and label a.

Coloring semigroup: identifying the coloring with the set of transformations $\mathcal{C} = \{R_a : a \in A\}$, we refer to the semigroup $S = \langle \mathcal{C} \rangle$, generated by $\mathcal{C}$, as the *coloring semigroup* for the given labelling.

Kernel: Any finite transformation semigroup S has a minimal ideal or *kernel,* which consists of the elements of minimal rank (see [5]). This common minimal rank is called the *rank of the kernel*.

Synchronizing instruction: Any transformation of rank one.

Rees Product: Let S be a coloring semigroup of a strongly connected digraph with kernel K. Then K is of the form $X \times G \times Y$ relative to a minimal idempotent e_0, say,

$$X = E(Ke_0)\,, \quad G = e_0 K e_0\,, \quad Y = E(e_0 K)$$

$E(\cdot)$ denoting "idempotents of", with product

$$(x_1, g_1, y_1)(x_2, g_2, y_2) := (x_1, g_1(y_1 x_2)g_2, y_2)$$

Sandwich function:

$$\phi\colon Y \times X \to G\,, \quad \phi(y, x) = yx$$

is fundamental in the structure of K. Recall that if $\phi(y, x) = e_0$ for all $(y, x) \in Y \times X$, then $X \times G \times Y$ is called a **direct product**.

Right group: If $X = E(Ke_0) = \{e_0\}$, a single idempotent, K is a right group.

When a coloring semigroup has a synchronizing (rank one) instruction, K can easily shown to be a right group.

The importance of right groups in the context of the road coloring problem has been shown in recent papers [2, 3] and our work continues to explore their rôle.

It follows from the results in [3] that if any coloring semigroup of a strongly connected aperiodic digraph generates a kernel that is a right group, then the digraph has some coloring semigroup that contains a synchronizing instruction.

References

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2 Solvability and right groups

Solvable Lie algebra: The main property of a solvable Lie algebra that we are using here is *Lie's Theorem* to the effect that (over an algebraically closed field) a solvable Lie algebra of matrices can be simultaneously upper-triangularized. Especially, Radjavi's Theorem on permutable traces is part of the inspiration behind the proof.

In general, we denote by $\mathfrak{g}(\cdot)$ the Lie algebra generated by transformations from a given set.

$\mathcal{L} = \mathfrak{g}(\mathcal{C})$: generated by the transformations in $\mathcal{C}$, is a subalgebra of $\mathfrak{g}(S)$ and is our main object of interest.

The main feature is that the generators $\mathcal{C}$ are simultaneously upper-triangularizable if and only if $\mathcal{L}$ is solvable.
In that case, we call the graph *Lie-solvable*.

2.1 Solvability implies Right Group

Lemma 2.1 *If $\mathcal{L}$ is solvable, then the kernel K is isomorphic to a Rees product semigroup that is a direct product.*

Theorem 2.2 *If $\mathcal{L}$ is solvable, then the kernel K is a right group.*

Combining this with the results of [3], we have

Corollary 2.3 If the Lie algebra $\mathcal{L}$ of a coloring semigroup is solvable, then there exists a coloring semigroup of the graph that contains a synchronizing instruction. In other words,

the road coloring conjecture holds
for Lie-solvable graphs

3 Lie algebra generated by idempotents

$\mathfrak{g} = \mathfrak{g}(x, y)$: where x and y are two idempotents

$u = x - y$ and $v = 1 - (x + y)$

satisfy the basic identities

$$\mathbf{uv + vu = 0} \quad \text{and} \quad \mathbf{u^2 + v^2 = 1}$$

Lie product:

$a \times b = (ab - ba)/2 = (1/2)\,[a, b]$.

Thus, $u \times v = uv = -v \times u$.

Center: of the associative algebra $\mathcal{A}(1,\ x,\ y)$ is generated by $\{1, u^2, v^2\}$.

4 Lie algebra of a completely simple semigroup

Take two idempotents from a finite, completely simple semigroup call them e and f.

We assume that they have neither the same partition nor the same range.

We know that ef is in the local group with the same partition as e and the same range as f.

Order: Let p be the order of ef in that group so that $(ef)^p$ is an idempotent.

ef and fe have the same order (in their respective groups)

We can find a spanning set, generic basis, of $3p+1$ elements.

4.1 Group generated by v's

v is invertible: In fact, we have

Denote idempotents $e' = (ef)^p$ and $f' = (fe)^p$. Then $v(e,f) = 1 - e - f$ and $v(e',f') = 1 - e' - f'$ satisfy $v(e,f)v(e',f') = 1$, i.e.,

$$v(e,f)^{-1} = v(e',f')$$

Proof: We have the 2×2 array

$$\begin{array}{c|c|c|} & R_1 & R_2 \\ \hline \pi_1 & f' & f \\ \hline \pi_2 & e & e' \\ \hline \end{array} \tag{1}$$

Recall that the columns form left-zero semigroups and the rows, right-zero semigroups. Now, multiplying out $(1 - e - f)(1 - e' - f')$ yields

$$1 - e' - f' - e + ee' + ef' - f + fe' + ff'$$

which simplifies down to 1 using the zero-properties just noted. $\square$

4.2 Levi-Malcev decomposition and oscillator subalgebra

Diagonalize the linear map $v_0\times$ acting on $\mathfrak{g}(e,f)$

The Levi-Malcev decomposition $\mathfrak{g}(e,f) = \mathcal{G} \oplus \mathcal{I}$ is completely described by the root-space decomposition:

the **semisimple part** $\mathcal{G}$
is isomorphic to a direct sum of $p-1$ copies of $\mathfrak{sl}(2)$

the **solvable radical** $\mathcal{I}$
corresponds to eigenvalues ± 1. Generically it is the four-dimensional oscillator algebra, $\mathfrak{osc}$.

Lie-solvable kernels and right groups

$\mathcal{L} = \mathfrak{g}(\mathcal{C})$**:** If the kernel is not a right group, then we can find e and f such that ef is not itself an idempotent, so that $p > 1$.
Thus $\mathcal{G}$ is nontrivial and $\mathcal{L}$ is not solvable.

In other words, $\mathcal{L}$ solvable implies that the kernel is a right group.

For right groups $\mathfrak{g}$ is a two-step solvable Lie algebra.

5 Examples

An interesting class of examples have a nontrivial sandwich function guaranteed for at least one coloring.
Let $V = \{1, 2, \ldots, 2k\}$ for any integer $k \geq 2$.

$R_1 = \{1, 3, \ldots, 2k-1\}$ and $R_2 = \{2, 4, \ldots, 2k\}$.
$\pi_1 = \{\{1, 2\}, \{3, 4\}, \ldots, \{2k-1, 2k\}\}$ is compatible with both R_1 and R_2.

In “transformation notation,”

$$r = [3, 3, 5, 5, \ldots, 2k-1, 2k-1, 1, 1]$$

Let $(i_1, i_2, \ldots, i_{k-1})$ be a permutation of the even integers $\{4, \ldots, 2k\}$ and define

$$b = [2, 4, i_1, 6, i_2, \ldots, 2k, i_{k-1}, 2]$$

π_2 is the partition induced from b.
It can be shown that these correspond to strongly connected, aperiodic digraphs. Furthermore, noting that this class of digraphs has uniform indegree as well as uniform outdegree **it follows from Kari [7] that the road coloring conjecture is true for this class of digraphs**.

5.1 Examples

There are two cases of the above construction for $k = 3$.
We will look at the associated kernel and some recolorings.

Notation. To denote the structure of a Lie algebra $\mathfrak{g}$ with Levi decomposition $\mathcal{G} \oplus \mathcal{I}$, we use the notation

$$\mathfrak{g} \sim \mathbf{d} \oplus \mathbf{n}/n_1$$

where $d = \dim \mathcal{G}$, $n = \dim \mathcal{I}$ and $n_1 = \dim[\mathcal{I}, \mathcal{I}]$.

$\bullet/n$ denotes a solvable algebra $\mathfrak{g}$ of dimension $n+1$, with $[\mathfrak{g}, \mathfrak{g}]$ n-dimensional abelian.

For example, $\bullet/1 \approx \mathfrak{aff}(2)$.

Example 1 $r = [3,3,5,5,1,1]$, $b = [2,4,4,6,6,2]$

The kernel has the shape

$$\begin{array}{ccc} & 135 & 246 \\ 12|34|56 & e_1 & e_2 \\ 16|23|45 & e_3 & e_4 \end{array}$$

Groups are $\approx C_3$, the cyclic group of order 3, so $p = 3$.

$$\mathcal{L} = \mathcal{K} \sim \mathbf{6} \oplus \mathbf{2}/1$$

i.e., $\mathcal{G}$ isomorphic to two copies of $\mathfrak{sl}(2)$ and $\mathcal{I} \approx \mathfrak{aff}(2)$.

Recolor to $r = [3, 3, 5, 6, 6, 2]$, $b = [2, 4, 4, 5, 1, 1]$, now $\mathcal{L} \sim \mathbf{8} \oplus \mathbf{4}/2$, and $\mathcal{K} \sim \mathbf{8} \oplus \bullet/2$.

The kernel has the shape

$$\begin{array}{cccc} & 14 & 25 & 36 \\ 123|456 & e_1 & e_2 & e_3 \\ 126|345 & e_4 & e_5 & e_6 \\ 156|234 & e_7 & e_8 & e_9 \end{array}$$

Groups are C_2's.

Denote, e.g., $[1 \times 5]$ the block of four cells with diagonal containing e_1 and e_5, with $\mathfrak{g}(1 \times 5)$ the corresponding Lie algebra.

Then $[1 \times 5]$ is a direct product, $p = 1$, i.e., $e_1 e_5 = e_2$, with $\mathfrak{g}(1 \times 5)$ an oscillator algebra.

While $[1 \times 6]$ has $p = 2$, with

$\mathfrak{g}(1 \times 6) \sim \mathbf{3} \oplus \bullet/1 \approx \mathfrak{sl}(2) \oplus \mathfrak{aff}(2)$.

Example 2. $r = [3, 3, 5, 5, 1, 1]$, $b = [2, 4, 6, 6, 4, 2]$
The kernel has the shape

$$\begin{array}{ccc} & 135 & 246 \\ 12|34|56 & e_1 & e_2 \\ 16|25|34 & e_3 & e_4 \end{array}$$

The local groups are now S_3 — symmetric groups.
Here $p = 2$,
$\mathcal{L} = \mathcal{K} \sim \mathbf{8} \oplus \bullet/4$, with $\mathbf{8} \approx \mathfrak{sl}(3)$.
The Lie algebra

$$\mathfrak{g}(1 \times 4) \sim \mathbf{3} \oplus \bullet/1 \approx \mathfrak{sl}(2) \oplus \mathfrak{aff}(2)$$

Recoloring, $r = [3, 4, 6, 5, 4, 1]$, $b = [2, 3, 5, 6, 1, 2]$.
The kernel is a right group, but $\mathcal{L} \sim \mathbf{8} \oplus \mathbf{13}/11$ is not solvable, the $\mathbf{8} \approx \mathfrak{sl}(3)$. The kernel has the shape, only one partition,

$$\begin{array}{cccccccccc} & 12 & 14 & 15 & 32 & 34 & 35 & 62 & 64 & 65 \\ 136|245 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 \end{array}$$

The Lie algebra generated by the idempotents $\mathfrak{g} \sim \bullet/4$, with $\mathcal{K} \sim 0 \oplus \mathbf{10}/8$.

6 Conclusion

Solvability:

What is an equivalent graph-theoretic condition?

Lie algebras: warrant further study in this context

Covering group: generated by the v-operators. What is its relation to the group $G = e_0 K e_0$ of the Rees product as well as the subgroup within G generated by the sandwich function?

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The original calculations of Lie algebras generated by coloring transformations were a part of the Master's Thesis of J. Gill [4], who observed that in every case that the Lie algebra was solvable the kernel was a right group.

GAP

The Lie algebra calculations were carried out using GAP.