Second Quantization and Recurrences

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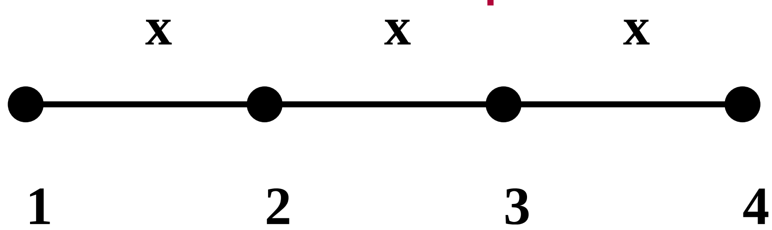
Via recurrences, we find the
matching polynomials of cyclically labelled
paths, cycles, and trees.
The technique is to use
trace formulas for matrices acting on the space of
symmetric tensors.

Special Session on Special Functions
and Orthogonal Polynomials
Annandale-on-Hudson, NY
9 October 2005

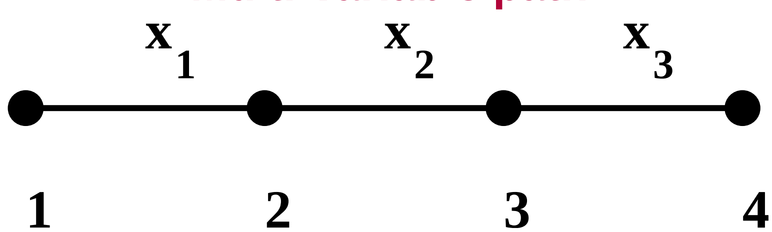
1 Matching polynomials

One-variable path



$$1 + 3x + x^2$$

Multi-variable path

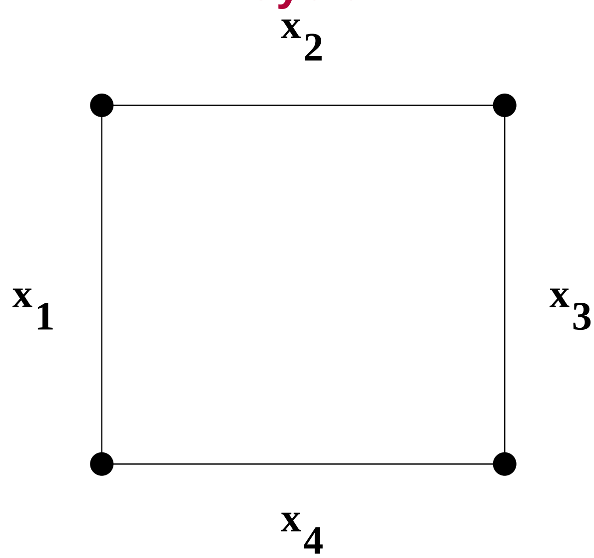


$$1 + x_1 + x_2 + x_3 + x_1 x_3$$

nc-function: ϕ_n is the sum of all *nonconsecutive* monomials in the variables $x_1, x_2, \ldots, x_n$.

Reciprocal-Chebyshev 2$^{\mathrm{nd}}$ kind: $\phi_{n-1}(x) = \sum_k \binom{n-k}{k} x^k$

Cycle

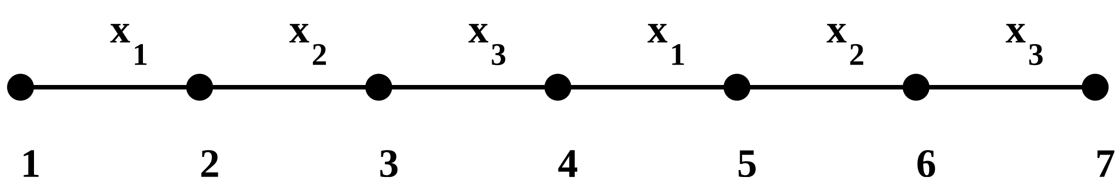


$$1 + x_1 + x_2 + x_3 + x_4 + x_1x_3 + x_2x_4$$

ncc-function: τ_n is the sum of all *nonconsecutive, cyclic* monomials in the variables $x_1, x_2, \ldots, x_n$.

Reciprocal-Chebyshev 1$^{\text{st}}$ kind: $\tau_n(x) = \sum_k \binom{n-k}{k} \frac{n}{n-k} x^k$

Multi-variable cyclic path



$$1 + 2x_1 + 2x_2 + 2x_3 + x_1^2 + 2x_1x_2 + 3x_1x_3$$
$$+x_2^2 + 2x_2x_3 + x_3^2 + x_1^2x_3 + 2x_1x_2x_3 + x_1x_3^2$$

This is the question

2 Recurrences and matrices

- **nc-Recurrence**

$$\phi_n = \phi_{n-1} + x_n \phi_{n-2}$$

The **nc**-function ϕ_n satisfies this recurrence with I.C.'s $\phi_{-1} = 1$, $\phi_0 = 1$.

Denoting by f_n and g_n the fundamental solutions to this recurrence, we have $\phi_n = f_n + g_n$.

- **Matrices**

$$X = X_n = \begin{pmatrix} g_{n-1} & f_{n-1} \\ g_n & f_n \end{pmatrix}$$

The **ncc**-function $\tau_n = g_{n-1} + f_n$ is the trace of X_n.

The matrix **factors** as

$$X = \begin{pmatrix} 0 & 1 \\ x_n & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ x_{n-1} & 1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ x_1 & 1 \end{pmatrix}$$

2.1 Tau-Delta recurrence

Any matrix element $\psi_N = \langle \mathbf{u}, X^N \mathbf{v} \rangle$, $\mathbf{u}, \mathbf{v} \in \mathbf{R^2}$, satisfies the **tau-Delta recurrence**

$$\psi_N = \tau\, \psi_{N-1} - \Delta\, \psi_{N-2}$$

where $\tau = \operatorname{tr} X$ and $\Delta = \det X = (-1)^n x_1 x_2 \cdots x_n$.

- **First fundamental solution**

$$G_N = \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N-k}{k} \tau^{N-2k} (-\Delta)^k$$

- **Generating function**

$$\frac{1}{\det(I - tX)} = \sum_{N=0}^{\infty} t^N\, G_N$$

Powers of X correspond to cyclic repetition of the initial path with n edges.

3 Second quantization and trace formulas

- **Symmetric representation** of a $d \times d$ matrix A

With $\mathbf{u} = (u_1, \ldots, u_d)^T$, $\mathbf{v} = (v_1, \ldots, v_d)^T$,

$$\mathbf{v} = A\mathbf{u}$$

For given homogeneous degree N, define

matrix elements by

$$v_1^{m_1} \cdots v_d^{m_d} = \sum_{n_1, \ldots, n_d} \left\langle \begin{matrix} m_1, \ldots, m_d \\ n_1, \ldots, n_d \end{matrix} \right\rangle_A u_1^{n_1} \cdots u_d^{n_d}$$

$$\mathbf{v}^{\mathbf{m}} = \sum_{\mathbf{n}} \left\langle \begin{matrix} \mathbf{m} \\ \mathbf{n} \end{matrix} \right\rangle_A \mathbf{u}^{\mathbf{n}}$$

This is a representation of the multiplicative semigroup of matrices. In other words, we have the

- **Homomorphism property**

$$\left\langle \begin{matrix} \mathbf{m} \\ \mathbf{n} \end{matrix} \right\rangle_{AB} = \sum_{\mathbf{k}} \left\langle \begin{matrix} \mathbf{m} \\ \mathbf{k} \end{matrix} \right\rangle_A \left\langle \begin{matrix} \mathbf{k} \\ \mathbf{n} \end{matrix} \right\rangle_B$$

3.1 Symmetric traces

- The action defined here on polynomials is equivalent to the action on symmetric tensor powers, as in classical invariant theory. See Fulton and Harris [Representation theory, a first course, pp. 472-5].

- **boson Fock space** over the d-dimensional vector space is the space of symmetric tensor powers.

- **Symmetric trace:** for fixed homogeneous degree N the *symmetric trace* of A in degree N

$$\mathrm{tr}^N_{\mathrm{Sym}}(A) = \sum_{|\mathbf{m}|=N} \left\langle \begin{matrix} \mathbf{m} \\ \mathbf{m} \end{matrix} \right\rangle_A$$

- **Symmetric trace theorem**

(See Springer [Invariant theory, LNM 585, pp. 51-2].)

$$\frac{1}{\det(I - tA)} = \sum_{N=0}^{\infty} t^N \mathrm{tr}^N_{\mathrm{Sym}}(A).$$

• **Tau-Delta recurrence revisited**

For G_N, the first fundamental solution to the τ-Δ recurrence, the Symmetric Trace Theorem says

$$\begin{aligned} G_N &= \mathrm{tr}^N_{\mathrm{Sym}}(X) = \sum_{|\mathbf{m}|=N} \left\langle \begin{matrix} \mathbf{m} \\ \mathbf{m} \end{matrix} \right\rangle_X \\ &= \sum_{|\mathbf{m}|=N} \left\langle \begin{matrix} \mathbf{m} \\ \mathbf{m} \end{matrix} \right\rangle_{\xi_n \xi_{n-1} \cdots \xi_1} \end{aligned}$$

By the Homomorphism Property, we calculate the matrix elements for each factor ξ_i.

• **Matrix elements** for $\xi_i = \begin{pmatrix} 0 & 1 \\ x_i & a_i \end{pmatrix}$. The mapping induced on polynomials is

$$v_1 = u_2\,, \quad v_2 = x_i\, u_1 + a_i\, u_2$$

And we find, for fixed homogeneous degree N,

$$\left\langle \begin{matrix} m \\ n \end{matrix} \right\rangle_{\xi_i} = \binom{N-m}{n} x_i^n\, a_i^{N-m-n}$$

4 Cyclic binomial identity

$$
\begin{aligned}
G_N &= \sum_{k_1,\ldots,k_n} \binom{N-k_2}{k_1}\binom{N-k_3}{k_2}\cdots\binom{N-k_n}{k_{n-1}}\binom{N-k_1}{k_n} \\
&\qquad \times x_1^{k_1}\cdots x_n^{k_n} a_1^{N-k_1-k_2} a_2^{N-k_2-k_3}\cdots a_n^{N-k_n-k_1} \\
&= \Delta^{N/2} U_N\left(\frac{\tau}{2\sqrt{\Delta}}\right) \\
&= \sum_{k=0}^{\lfloor N/2 \rfloor} \binom{N-k}{k} \tau^{N-2k}(-\Delta)^k \\
&= \sum_{m,k} \binom{m}{k}\binom{N-m}{m-k} f_n^{N-2m+k} g_{n-1}^k (f_{n-1}g_n)^{m-k}
\end{aligned}
$$

where U_N denotes the Chebyshev polynomial of the second kind.

Recall f_n and g_n are the fundamental solutions to the initial n-step recurrence.

5 Comments

- **Special functions interest**

n=2 finite ${}_2F_1$ summation or *Chu-Vandermonde* sum

n=3 gives ${}_3F_2$ *Pfaff-Saalschütz* sum

n$\geq$4 gives a multivariate summation formula that requires further investigation

- **Matching polynomials**

$G_N + (\phi_n - \tau_n)G_{N-1}$ is the matching polynomial for the N-fold repeated path of length n

$2\,\Delta^{N/2}\,T_N\left(\frac{\tau}{2\sqrt{\Delta}}\right)$ is for the corresponding cycle.

Formulas for trees.

6 Conclusion

● **Second quantization of a recurrence** which is the periodic extension [constant coefficients] of a given recurrence [non-constant coefficients] yields identities in the underlying variables by interpreting the fundamental solution in various ways.

● **Hierarchy of hierarchies of identities** since for fixed r, an r-step recurrence gives a hierarchy of identities.
Now vary r.

● **Relation** with mathematical objects such as multivariate Chebyshev polynomials?