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Spanning Forests of a Digraph and Their Applications[1]

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Abstract—We study spanning diverging forests of a digraph and related matrices. It is shown that the normalized matrix of out forests of a digraph coincides with the transition matrix in a specific observation model for Markov chains related to the digraph. Expressions are given for the Moore-Penrose generalized inverse and the group inverse of the Kirchhoff matrix. These expressions involve the matrix of maximum out forests of the digraph. Every matrix of out forests with a fixed number of arcs and the normalized matrix of out forests are represented as polynomials of the Kirchhoff matrix; with the help of these identities, new proofs are given for the matrix-forest theorem and some other statements. A connection is specified between the forest dimension of a digraph and the degree of an annihilating polynomial for the Kirchhoff matrix. Some accessibility measures for digraph vertices are considered. These are based on the enumeration of spanning forests.

1. INTRODUCTION

Directed graphs provide a simple and universal tool to model connection structures. It is not accidental that the first systematic monograph in the theory of digraphs [1] was titled "Structural Models: An Introduction to the Theory of Directed Graphs." Digraphs frequently serve to model processes that can proceed in the direction of arcs. Physical transference, service, control, transmission of influences, ideas, innovations, and diseases are examples of such processes. If a process can start from a number of vertices and ends with the inclusion of all vertices, then the process can be modelled by the family of out forests (i.e., spanning diverging forests) of the digraph. The enumeration of all out forests allows one to determine the typical roles of the vertices in the process: one vertex is a typical starting point, another vertex is a typical intermediate point, some vertex is a typical terminating point of the process, etc. If an initial (weighted) digraph imposes some measure on the said processes, then the "role profile" of each vertex can be expressed numerically. Moreover, an exact answer can be given to the following important question: how likely is it that the process initiated at vertex j arrives at vertex i. It is not surprising that out forests of a digraph turn out to be closely related with Markov chains realizable on the digraph.

The study of out forests has been started in [2]. Generally, they were given less attention in the literature, than that given to spanning diverging trees (out arborescences), which exist only for a narrow class of digraphs. We mention in this connection [3–11], where still undirected forests were considered in most cases. The maximum out forests (i.e., out forests with the greatest possible number of arcs) of a digraph were studied in [12,13]. It was established that the normalized matrix of such forests coincides with the matrix of limiting probabilities of every Markov chain *related* to the given digraph. Some results on spanning forests of directed and undirected multigraphs were given in [14,15].

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