SPANNING FORESTS AND THE GOLDEN RATIO*

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Abstract

For a graph G, let f_{ij} be the number of spanning rooted forests in which vertex j belongs to a tree rooted at i. In this paper, we show that for a path, the f_{ij}'s can be expressed as the products of Fibonacci numbers; for a cycle, they are products of Fibonacci and Lucas numbers. The *doubly stochastic graph matrix* is the matrix $F = \frac{(f_{ij})_{n\times n}}{f}$, where f is the total number of spanning rooted forests of G and n is the number of vertices in G. F provides a proximity measure for graph vertices. By the matrix forest theorem, $F^{-1} = I + L$, where L is the Laplacian matrix of G. We show that for the paths and the so-called T-caterpillars, some diagonal entries of F (which provide a measure of the self-connectivity of vertices) converge to ϕ^{-1} or to $1 - \phi^{-1}$, where ϕ is the golden ratio, as the number of vertices goes to infinity. Thereby, in the asymptotic, the corresponding vertices can be metaphorically considered as "golden introverts" and "golden extroverts," respectively. This metaphor is reinforced by a Markov chain interpretation of the doubly stochastic graph matrix, according to which F equals the overall transition matrix of a random walk with a random number of steps on G.

Keywords: Doubly stochastic graph matrix; Matrix forest theorem; Fibonacci numbers; Laplacian matrix; Vertex-vertex proximity; Spanning forest; Golden ratio

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1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$, $|V| = n$, and edge set $E = E(G)$. Suppose that $n \geq 2$.

A *spanning rooted forest of* G is any spanning acyclic subgraph of G with a single vertex (a *root*) marked in each tree.

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