SPANNING FORESTS AND THE GOLDEN RATIO*

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Abstract

For a graph G, let f_{ij} be the number of spanning rooted forests in which vertex j belongs to a tree rooted at i. In this paper, we show that for a path, the f_{ij}'s can be expressed as the products of Fibonacci numbers; for a cycle, they are products of Fibonacci and Lucas numbers. The *doubly stochastic graph matrix* is the matrix $F = \frac{(f_{ij})_{n\times n}}{f}$, where f is the total number of spanning rooted forests of G and n is the number of vertices in G. F provides a proximity measure for graph vertices. By the matrix forest theorem, $F^{-1} = I + L$, where L is the Laplacian matrix of G. We show that for the paths and the so-called T-caterpillars, some diagonal entries of F (which provide a measure of the self-connectivity of vertices) converge to ϕ^{-1} or to $1-\phi^{-1}$, where ϕ is the golden ratio, as the number of vertices goes to infinity. Thereby, in the asymptotic, the corresponding vertices can be metaphorically considered as "golden introverts" and "golden extroverts," respectively. This metaphor is reinforced by a Markov chain interpretation of the doubly stochastic graph matrix, according to which F equals the overall transition matrix of a random walk with a random number of steps on G.

Keywords: Doubly stochastic graph matrix; Matrix forest theorem; Fibonacci numbers; Laplacian matrix; Vertex-vertex proximity; Spanning forest; Golden ratio

AMS Classification: 05C50, 05C05, 05C12, 15A51, 11B39, 60J10

1 Introduction

Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$, $|V| = n$, and edge set $E = E(G)$. Suppose that $n \geq 2$.

A *spanning rooted forest of* G is any spanning acyclic subgraph of G with a single vertex (a *root*) marked in each tree.

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Let $f_{ij} = f_{ij}(G)$ be the number of spanning rooted forests of G in which vertices i and j belong to the same tree rooted at i. The matrix $(f_{ij})_{n\times n}$ is the *matrix of spanning rooted forests of* G. Let $f = f(G)$ be the total number of spanning rooted forests of G.

The matrix $F = \frac{(f_{ij})_{n\times n}}{f}$ is referred to as the *doubly stochastic graph matrix* [14, 15, 24, 23] or the *matrix of relative connectivity via forests*. By the matrix forest theorem [7, 8, 5],

$$F^{-1} = I + L \tag{1}$$

and

$$f = \det(I + L), \tag{2}$$

where L is the *Laplacian matrix* of G, i.e. $L = D - A$, A being the adjacency matrix of G and D the diagonal matrix of vertex degrees of G. Most likely, the matrix $(I + L)^{-1} = F$ was first considered in [11]. Chaiken [4] used the matrix $\mathrm{adj}(I + L) = (f_{ij})_{n\times n}$ for coordinatizing linking systems of strict gammoids. The (i, j) entry of F can be considered as a measure of proximity between vertices i and j in G; the (i, i) entry measures the self-connectivity of vertex i.

A *path* is a connected graph in which two vertices have degree 1 and the remaining vertices have degree 2. Let P_n be the path with $V(P_n) = \{1, 2, \ldots, n\}$ and $E(P_n) = \{(1, 2), (2, 3), \ldots, (n - 1, n)\}$, see Fig. 1(a).

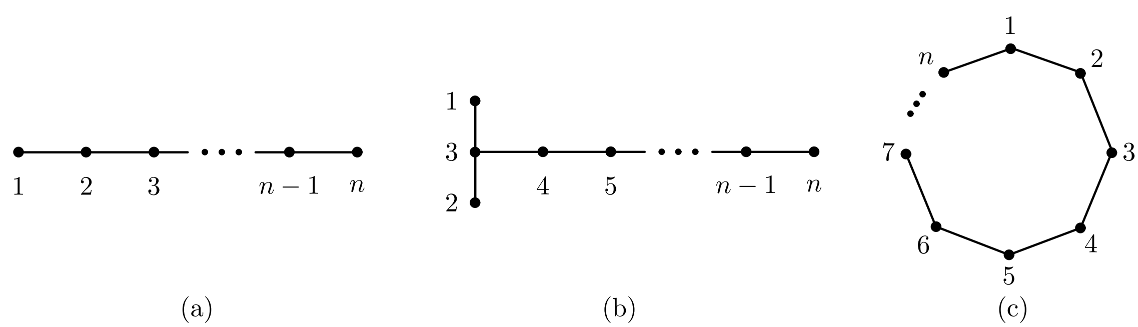

Figure 1: (a) the path P_n; (b) the T-caterpillar T_n; (c) the cycle C_n.

All spanning rooted forests of P_4 and the spanning rooted forests in which vertex 1 belongs to a tree rooted at vertex 2 are shown in Fig. 2, where thick dots denote roots.

The matrix F for P_4 is

$$F(P_4) = \frac{(f_{ij})}{f} = \frac{1}{21}\begin{bmatrix} 13 & 5 & 2 & 1 \\ 5 & 10 & 4 & 2 \\ 2 & 4 & 10 & 5 \\ 1 & 2 & 5 & 13 \end{bmatrix}.$$

Let T_n be the graph obtained from P_n by replacing the edge $(1, 2)$ with $(1, 3)$: $V(T_n) = \{1, 2, \ldots, n\}$ and $E(T_n) = \{(1, 3), (2, 3), (3, 4), \ldots, (n - 1, n)\}$, see Fig. 1(b). We call T_n a *T-caterpillar*.

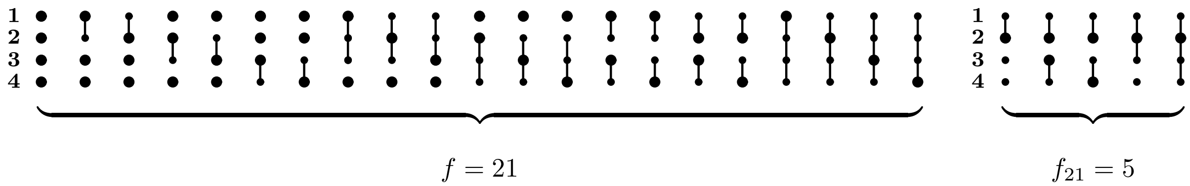

Figure 2: The spanning rooted forests in P_4 and the forests where 1 is in a tree rooted at 2.

Let C_n, $n \geq 3$, be the *cycle* on n vertices: $V(C_n) = \{1, 2, \ldots, n\}$ and $E(C_n) = \{(1,2), (2,3), \ldots, (n-1, n), (n, 1)\}$, Fig. 1(c).

By $(\Phi_i)_{i=0,1,2,\ldots} = (0, 1, 1, 2, 3, 5, \ldots)$ we denote the Fibonacci numbers. Sometimes, it is convenient to consider the subsequences of Fibonacci numbers with odd and even subscripts separately:

$$\begin{aligned} \Phi'_i &= \Phi_{2i-1}, \quad i = 1, 2, \ldots; \\ \Phi''_i &= \Phi_{2i}, \qquad i = 0, 1, 2, \ldots. \end{aligned}$$

In Section 2 we study the spanning rooted forests in paths, cycles, and T-caterpillars, in Section 3 the results are interpreted in terms of vertex-vertex proximity, and Sections 4 and 5 present interpretations of the doubly stochastic graph matrix in terms of random walks and information dissemination, respectively.

2 Spanning rooted forests in paths, cycles, and T-caterpillars

Theorem 1 *Let G be a path, $G = P_n$. Then $f = \Phi''_n$ and $f_{ij} = \Phi'_{\min(i,j)} \cdot \Phi'_{n+1-\max(i,j)}$ for all $i, j = 1, \ldots, n$.*

The number $f(G)$ of spanning rooted forests in any graph G is equal to the number of spanning trees in the graph G^{+1}, which is G augmented by a “hub” vertex adjacent to every vertex of G. Indeed, a bijection between the spanning rooted forests of G and spanning trees of G^{+1} is established by connecting every root of every spanning rooted forest to the “hub” by an edge. If $G = P_n$, then G^{+1} is a fan graph sometimes also called a “terminated ladder.” The fact that the number of spanning trees in this G^{+1} equals Φ''_n is familiar to electrical network theorists ([18], cf. [16, 17, 1]). Among others, it was obtained by Hilton [12]. Myers [20] proved this using the notion of weighted composition; in [3] this fact was derived using Chebyshev polynomials. Our aim is to give a direct combinatorial proof of Theorem 1 in order to fully clarify the recurrence structure of spanning rooted forests in a path. Here, a proof of $f(P_n) = \Phi''_n$ is integrated with a proof of the expression for f_{ij} given in Theorem 1.

For any graph G, $\mathcal{F}(G)$ will denote the set of spanning rooted forests of G.

Proof of Theorem 1. Let $\mathcal{F}^m = \mathcal{F}(P_m)$, let $f(m) = |\mathcal{F}^m|$, $m = 1, 2, \ldots$. Then for every $k \geq 1$,

$$f(k+1) = |\mathcal{F}^{k+1}_{(1,2)}| + |\mathcal{F}^{k+1}_{\overline{(1,2)}}|, \tag{3}$$

where $\mathcal{F}^m_{(1,2)} = \{\mathsf{F} \in \mathcal{F}^m \mid (1,2) \in E(\mathsf{F})\}$ and $\mathcal{F}^m_{\overline{(1,2)}} = \mathcal{F}^m \smallsetminus \mathcal{F}^m_{(1,2)}$.

Let $\mathcal{F}^m_* = \mathcal{F}_*(P_m) = \{\mathsf{F} \in \mathcal{F}^m \mid 1 \text{ is a root in } \mathsf{F}\}$. Then in (3)

$$|\mathcal{F}^{k+1}_{\overline{(1,2)}}| = |\mathcal{F}^k| \quad\text{and}\quad |\mathcal{F}^{k+1}_{(1,2)}| = |\mathcal{F}^{k+1}_*|. \tag{4}$$

Indeed, a bijection between $\mathcal{F}^{k+1}_{\overline{(1,2)}}$ and $\mathcal{F}^k$ can be established by the restriction of each $\mathsf{F} \in \mathcal{F}^{k+1}_{\overline{(1,2)}}$ to the vertex subset $\{2, \ldots, k\}$; a bijection between $\mathcal{F}^{k+1}_{(1,2)}$ and $\mathcal{F}^{k+1}_*$ is established as follows: for every $\mathsf{F} \in \mathcal{F}^{k+1}_{(1,2)}$ obtain F' by putting $\mathsf{F}' = \mathsf{F}$ if 1 is a root in F and by removing edge $(1,2)$ and marking vertex 1 as a root, otherwise. Then $\mathsf{F}' \in \mathcal{F}^{k+1}_*$ and this correspondence is one-to-one.

By (3) and (4),

$$f(k+1) = f(k) + f^*(k+1) \quad k = 1, 2, \ldots, \tag{5}$$

where $f^*(m) = |\mathcal{F}^m_*|$. Let $\mathcal{F}^m_{(*1,2)} = \mathcal{F}^m_* \cap \mathcal{F}^m_{(1,2)}$, $m = 1, 2, \ldots$. Using (4) we obtain

$$f^*(k+1) = |\mathcal{F}^{k+1}_{\overline{(1,2)}}| + |\mathcal{F}^{k+1}_{(*1,2)}| = |\mathcal{F}^k| + |\mathcal{F}^k_*| = f(k) + f^*(k), \quad k = 1, 2, \ldots. \tag{6}$$

Here, a bijection between $\mathcal{F}^{k+1}_{(*1,2)}$ and $\mathcal{F}^k_*$ is established by coalescing vertex 2 with the root 1 and collapsing edge $(1,2)$.

Observe now that $f(1) = 1 = \Phi''_1$ and $f^*(1) = 1 = \Phi'_1$. By (5) and (6), $f(k)$ and $f^*(k)$ satisfy the same recurrence relations as Φ''_k and Φ'_k do, respectively. Therefore

$$f(k) = \Phi''_k \quad\text{and}\quad f^*(k) = \Phi'_k, \quad k = 1, 2, \ldots. \tag{7}$$

Thus, $f = f(n) = \Phi''_n$. To find f_{ij}, $i, j = 1, \ldots, n$, observe that f_{ij} counts the spanning rooted forests that contain the i–j path rooted at i. To obtain a spanning rooted forest, this path can be completed on the subset of vertices $\{1, \ldots, \min(i,j)\}$ in $f^*(\min(i,j))$ ways and on the subset of vertices $\{\max(i,j), \ldots, n\}$ in $f^*(n+1-\max(i,j))$ ways. Since the ways of these types are all compatible, (7) implies that $f_{ij} = \Phi'_{\min(i,j)} \cdot \Phi'_{n+1-\max(i,j)}$. $\square$

Theorem 1 as well as Theorem 2 below can also be proved algebraically by means of the matrix forest theorem (Eqs. (1) and (2)). We present combinatorial proofs here, since they are a bit more illuminating. However, Theorem 3 below is proved algebraically.

Theorem 2 *Let G be a cycle, $G = C_n$ with $n \geq 3$. Then $f = \Phi'_n + \Phi'_{n+1} - 2$ and $f_{ij} = \Phi''_{|j-i|} + \Phi''_{n-|j-i|}$, $i, j = 1, \ldots, n$.*

For $G = C_n$, the augmented graph G^{+1} mentioned above is the wheel on $n+1$ vertices. The fact that the number of spanning trees in the wheel is $\Phi'_n + \Phi'_{n+1} - 2$ is due to Sedláček [21] and Myers [19]. Myers [20] proved this using identities involving weighted compositions;

the proof by Benjamin and Yerger [2] is based on counting imperfect matchings. A useful tool for solving such problems is Chebyshev polynomials, see [17, 3, 25]. Our proof of the identity $f(C_n)=\Phi'_n+\Phi'_{n+1}-2$ presented here for completeness is based on relations between forests found before. The proof of Theorem 2 relies on the following lemma.

Lemma 1 *For $n\ge 2$, let $\mathcal{F}^n_{**}=\{\mathsf{F}\in\mathcal{F}(P_n)\,|\,1$ and n are roots in $\mathsf{F}\}$. Then $|\mathcal{F}^n_{**}|=\Phi''_{n-1}$.*

Proof. A bijection between $\mathcal{F}^n_{**}$ and the set $\mathcal{F}(P_{n-1})$ of spanning rooted forests in P_{n-1} can be established as follows. For every $\mathsf{F}\in\mathcal{F}(P_{n-1})$ define F' as the spanning subgraph of P_n whose roots satisfy two conditions:

(1) vertex i is a root in F' iff $[i=1$ or $i=n$ or $(i-1,i)\notin E(\mathsf{F})]$;

(2) $(i,i+1)\in E(\mathsf{F}')$ iff i is not a root in F.

In this case, F' is a spanning rooted forest of P_n. Indeed, if one assumes that some tree in F' has no root or has more than one root, then this would imply the presence of a tree with more than one root or no root in F, respectively. Furthermore, every $\mathsf{F}'\in\mathcal{F}^n_{**}$ has a pre-image in $\mathcal{F}(P_{n-1})$, and this correspondence is by definition one-to-one.

Consequently, by (7), $|\mathcal{F}^n_{**}|=|\mathcal{F}(P_{n-1})|=\Phi''_{n-1}$. $\square$

Proof of Theorem 2. Let $\mathcal{F}^m_{ij}$ be the set of spanning rooted forests of C_m in which j belongs to a tree rooted at i. Observe that

$$f_{ij}=\left|\mathcal{F}^n_{i\frown j}\right|+\left|\mathcal{F}^n_{i\smile j}\right|, \tag{8}$$

where $\mathcal{F}^n_{i\frown j}=\{\mathsf{F}\in\mathcal{F}^n_{ij}\,|\,(n,1)\notin E(\mathsf{F})\}$ and $\mathcal{F}^n_{i\smile j}=\{\mathsf{F}\in\mathcal{F}^n_{ij}\,|\,(n,1)\in E(\mathsf{F})\}$.

We now show that $\left|\mathcal{F}^n_{i\smile j}\right|=\Phi''_{|j-i|}$. Every forest in $\mathcal{F}^n_{i\smile j}$ contains the path $P_{n+1-|j-i|}$ formed by the vertices in the sequence $(\max(i,j),\dots,n,1,\dots,\min(i,j))$ and the edges between the neighboring elements in this sequence. The ways of completing this path to obtain a spanning rooted forest in C_n can be put into a one-to-one correspondence with the elements of the set $\mathcal{F}^{|j-i|+1}_{**}$ defined in Lemma 1. Indeed, linking each $\tilde{\mathsf{F}}\in\mathcal{F}^{|j-i|+1}_{**}$ with $P_{n+1-|j-i|}$ by replacing the vertices 1 and $|j-i|+1$ of $\tilde{\mathsf{F}}$ with vertices i and j of $P_{n+1-|j-i|}$, respectively, produces a spanning rooted forest of C_n, and every spanning rooted forest of C_n can be uniquely obtained in this manner. Thus by Lemma 1, $\left|\mathcal{F}^n_{i\smile j}\right|=|\mathcal{F}^{|j-i|+1}_{**}|=\Phi''_{|j-i|}$. Similarly, $\left|\mathcal{F}^n_{i\frown j}\right|=|\mathcal{F}^{n+1-|j-i|}_{**}|=\Phi''_{n-|j-i|}$. Therefore by (8) it follows that $f_{ij}=\Phi''_{|j-i|}+\Phi''_{n-|j-i|}$.

Let $\mathcal{F}^m=\mathcal{F}(C_m)$, $m\ge 3$. Then

$$f(C_n)=|\mathcal{F}_{(\overline{1,n})}|+|\mathcal{F}_{(1,n)*}|+|\mathcal{F}_{*(1,n)}|, \tag{9}$$

where

$$\begin{aligned}
\mathcal{F}_{(\overline{1,n})} &= \{\mathsf{F}\in\mathcal{F}^m\,|\,(1,n)\notin E(\mathsf{F})\},\\
\mathcal{F}_{(1,n)*} &= \{\mathsf{F}\in\mathcal{F}^m\,|\,(1,n)\in E(\mathsf{F})\text{ and the path joining 1 with the root contains } n\},\\
\mathcal{F}_{*(1,n)} &= \{\mathsf{F}\in\mathcal{F}^m\,|\,(1,n)\in E(\mathsf{F})\text{ and the path joining } n \text{ with the root contains } 1\}.
\end{aligned}$$

Obviously, $|\mathcal{F}_{(\overline{1,n})}|=|\mathcal{F}(P_n)|$, so, by Theorem 1, $|\mathcal{F}_{(\overline{1,n})}|=\Phi''_n$. Consider any $\mathsf{F}\in\mathcal{F}_{(1,n)*}$. Removing $(1,n)$ from $E(\mathsf{F})$ and marking 1 as a root produces a forest $\mathsf{F}'\in\mathcal{F}_*(P_n)$, where

$\mathcal{F}_*(P_n) = \{\mathsf{F} \in \mathcal{F}(P_n) \mid 1 \text{ is a root in } \mathsf{F}\}$ was defined in the proof of Theorem 1. All elements of $\mathcal{F}_*(P_n)$ can be obtained in this way, except for the whole path P_n rooted at 1. That is why $|\mathcal{F}_{(1,n)*}| = |\mathcal{F}_*(P_n)| - 1$ and, by (7), $|\mathcal{F}_{(1,n)*}| = \Phi'_n - 1$. Similarly, $|\mathcal{F}_{*(1,n)}| = \Phi'_n - 1$. Substituting this in (9) provides

$$f(C_n) = \Phi''_n + 2(\Phi'_n - 1) = \Phi'_n + \Phi'_{n+1} - 2. \qquad \square$$

Now recall that

$$\Lambda_i = \Phi_{i-1} + \Phi_{i+1},$$

where $\Phi_{-1} = 1$, are the Lucas numbers: $(\Lambda_i)_{i=0,1,2,\ldots} = (2, 1, 3, 4, 7, 11, 18, 29, 47, \ldots)$, see, e.g., [13]. The Lucas numbers satisfy the same recurrence as the Fibonacci numbers do:

$$\Lambda_i + \Lambda_{i+1} = \Lambda_{i+2}, \quad i = 0, 1, 2, \ldots,$$

but some other properties of the Lucas numbers are even more elegant than those of the Fibonacci numbers.

By Theorem 2, $f(C_n) = \Lambda_{2n} - 2$. The numbers of forests in a cycle can also be expressed via smaller Fibonacci and Lucas numbers, viz. Corollary 1 holds.[1]

Corollary 1 *Let G be a cycle, $G = C_n$ with $n \geq 3$. Then*

$$f = \begin{cases} \Lambda_n^2, & n = 2k-1 \\ 5\Phi_n^2, & n = 2k \end{cases}; \qquad f_{ij} = \begin{cases} \Phi_t \Lambda_n, & n = 2k-1 \\ \Lambda_t \Phi_n, & n = 2k \end{cases}, \quad i, j = 1, \ldots, n,$$

where $t = \big|n - 2\,|j - i|\big|$.

Corollary 1 is derived from Theorem 2 by means of classical identities involving Fibonacci and Lucas numbers. It provides a simple expression for the entries of the doubly stochastic graph matrix $F = \frac{(f_{ij})}{f}$ of C_n:

Corollary 2 *The entries of the doubly stochastic matrix $F = \frac{(f_{ij})}{f}$ of C_n $(n \geq 3)$ are:*

$$\frac{f_{ij}}{f} = \begin{cases} \Phi_t / \Lambda_n, & n = 2k-1 \\ \Lambda_t / 5\Phi_n, & n = 2k \end{cases}, \quad i, j = 1, \ldots, n, \tag{10}$$

where $t = \big|n - 2\,|j - i|\big|$.

In the expression (10), for every row of F, the numerators make up a segment of a fixed symmetric two-sided sequence: for odd n this sequence is $(\ldots, 34, 13, 5, 2, 1, 1, 2, 5, 13, 34, \ldots)$; for even n it is $(\ldots, 47, 18, 7, 3, 2, 3, 7, 18, 47, \ldots)$. Thereby the ratio of two corresponding elements of F is the same for all n of the same parity.

Regarding the T-caterpillars, we are mainly interested in the total number f of spanning rooted forests and the diagonal entries f_{33} and f_{nn} of the matrix of spanning rooted forests.

[1] A knot theory interpretation of the squareness of $\Lambda_{2n} - 2$ when n is odd can be found in [22].

Theorem 3 *Let G be a T-caterpillar, $G = T_n$. Then $f = 4\Phi'_{n-1}$, $f_{33} = 4\Phi'_{n-2}$, and $f_{nn} = 4\Phi''_{n-2}$.*

Proof. Observe that $\Phi''_0 = 0$, $\Phi''_1 = 1$, and for $i = 1, 2, \dots,$

$$\begin{aligned} \Phi''_{i+1} &= \Phi_{2i+2} = \Phi_{2i} + \Phi_{2i+1} = \Phi_{2i} + \Phi_{2i} + \Phi_{2i-1} \\ &= 2\Phi_{2i} + \Phi_{2i} - \Phi_{2i-2} = 3\Phi''_i - \Phi''_{i-1}. \end{aligned} \tag{11}$$

Similarly, $\Phi'_1 = 1$ and for $i = 1, 2, \dots,$

$$\Phi'_{i+1} = \begin{cases} 3\Phi'_i - \Phi'_{i-1}, & i > 1 \\ 2\Phi'_i, & i = 1 \end{cases}. \tag{12}$$

For a T-caterpillar,

$$I + L = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \tag{13}$$

Equations $F(I + L) = I$ (see (1)) and (13) imply

$$\begin{aligned} 2f_{n1} - f_{n3} &= 0, \\ 2f_{n2} - f_{n3} &= 0, \\ -f_{n1} - f_{n2} + 4f_{n3} - f_{n4} &= 0, \\ -f_{n3} + 3f_{n4} - f_{n5} &= 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots& \\ -f_{n,n-2} + 3f_{n,n-1} - f_{nn} &= 0, \\ -f_{n,n-1} + 2f_{nn} &= f. \end{aligned} \tag{14}$$

Note that $f_{n1} = f_{n2} = 2 = 2\Phi''_1$, since T_n has exactly two spanning rooted forests where vertex 1 belongs to a tree rooted at n, one of them being T_n with root n, the other T_n with edge $(2, 3)$ deleted and roots n and 2. Then, by (14), $f_{n3} = 2f_{n1} = 4 = 4\Phi''_1$. Consequently, using (14), (11) and induction, we obtain that

$$\begin{aligned} f_{n4} &= 4f_{n3} - f_{n1} - f_{n2} = 12 = 4(3\Phi''_1 - \Phi''_0) = 4\Phi''_2, \\ f_{n5} &= 3f_{n4} - f_{n3} = 4(3\Phi''_2 - \Phi''_1) = 4\Phi''_3, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ f_{nn} &= 3f_{n,n-1} - f_{n,n-2} = 4(3\Phi''_{n-3} - \Phi''_{n-4}) = 4\Phi''_{n-2}. \end{aligned} \tag{15}$$

From the last equation of (14), $f = 2f_{nn} - f_{n,n-1} = 4(2\Phi''_{n-2} - \Phi''_{n-3}) = 4(\Phi''_{n-2} + \Phi'_{n-2}) = 4\Phi'_{n-1}$. It remains to show that $f_{33} = 4\Phi'_{n-2}$. Eqs. (1) and (13) imply

$$\begin{aligned} -f_{33} + 3f_{34} - f_{35} &= 0, \\ -f_{34} + 3f_{35} - f_{36} &= 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ -f_{3,n-2} + 3f_{3,n-1} - f_{3n} &= 0, \\ -f_{3,n-1} + 2f_{3n} &= 0. \end{aligned} \tag{16}$$

Since $f_{3n} = f_{n3} = 4 = 4\Phi'_1$, from (16) and (12) we have that $f_{3,n-1} = 4\Phi'_2$ and, by induction, $f_{33} = 4\Phi'_{n-2}$. □

3 "Golden introverts" and "golden extroverts"

In [7, 8, 9] (see also [14, 15, 6]) $F = \frac{(f_{ij})_{n\times n}}{f}$ was studied as a matrix of vertex-vertex proximity. For every graph G, F is a positive definite doubly stochastic matrix and $\frac{f_{ij}}{f}$ measures the relative strength of connections between vertices i and j in G. This proximity measure was referred to as the *relative connectivity via forests*. For some additional applications of F we refer to [11, 10].

It turns out [15, Theorem 2] that for every pair of vertices i and j such that $j \neq i$, $f_{ij} \leq f_{ii}/2$; $\frac{f_{ii}}{f}$ can be considered as a measure of self-connectivity of vertex i. By [14, Corollary 7], $\frac{f_{ii}}{f} \geq (1 + d_i)^{-1}$, where d_i is the degree of vertex i.

A vertex i can be called "an introvert" if $\frac{f_{ii}}{f} > 0.5$ (or, equivalently, $f_{ii} > \sum_{j\neq i} f_{ij}$) and "an extrovert" if $\frac{f_{ii}}{f} < 0.5$ (equivalently, $f_{ii} < \sum_{j\neq i} f_{ij}$). The complete graph on three vertices provides an example of the boundary case where $\frac{f_{ii}}{f} = 0.5$ and $f_{ii} = \sum_{j\neq i} f_{ij}$ for every vertex i.

Proposition 1 *Let ϕ be the golden ratio, $\phi = \frac{\sqrt{5}+1}{2}$. Then*
(i) For the paths P_n, $\lim\limits_{n\to\infty} \frac{f_{11}}{f} = \phi^{-1}$;
(ii) For the T-caterpillars T_n, $\lim\limits_{n\to\infty} \frac{f_{nn}}{f} = \phi^{-1}$ and $\lim\limits_{n\to\infty} \frac{f_{33}}{f} = 1 - \phi^{-1}$.

Proof. By Theorem 1 and Binet's Fibonacci number formula, for the paths P_n, $\lim\limits_{n\to\infty}(f_{11}/f) = \lim\limits_{n\to\infty}(\Phi'_n/\Phi''_n) = \phi^{-1}$. By Theorem 3, for the T-caterpillars T_n, $\lim\limits_{n\to\infty}(f_{nn}/f) = \lim\limits_{n\to\infty}(4\Phi''_{n-2}/4\Phi'_{n-1}) = \phi^{-1}$ and $\lim\limits_{n\to\infty}(f_{33}/f) = \lim\limits_{n\to\infty}(4\Phi'_{n-2}/4\Phi'_{n-1}) = \phi^{-2} = 1 - \phi^{-1}$. □

Corollary 3 *Let ϕ be the golden ratio, $\phi = \frac{\sqrt{5}+1}{2}$. Then*
(i) For the paths P_n, $\lim\limits_{n\to\infty} \frac{f_{11}}{\sum\limits_{i\neq 1} f_{1i}} = \phi$;
(ii) For the T-caterpillars T_n, $\lim\limits_{n\to\infty} \frac{f_{nn}}{\sum\limits_{i\neq n} f_{ni}} = \phi$ and $\lim\limits_{n\to\infty} \frac{\sum\limits_{i\neq 3} f_{3i}}{f_{33}} = \phi$.

Corollary 3 follows from Proposition 1 and the fact that F is stochastic.

It can be shown that (ii) of Proposition 1 and (ii) of Corollary 3 remain true for the graphs resulting from T-caterpillars by the addition of edge $(1, 2)$.

In accordance with Corollary 3, as $n \to \infty$, vertices 1 and n in a path and vertex n in a T-caterpillar tend to be "golden introverts" (named after the golden ratio), whereas vertex 3 in a T-caterpillar tends to become a "golden extrovert." This provides a kind of sociological interpretation of Corollary 3.

4 A random walk interpretation of the doubly stochastic graph matrix

To better comprehend what exactly the results of the previous section mean, consider a random walk interpretation of the doubly stochastic graph matrix.

For a graph G, consider any Markov chain whose states are the vertices of G, $\{1, 2, \ldots, n\}$, and the probabilities of all $i \to j$ transitions with $i \neq j$ are proportional[2] to the corresponding elements of the adjacency matrix of G:

$$p_{ij} = \varepsilon a_{ij}, \quad i, j = 1, \ldots, n, \;\; i \neq j. \tag{17}$$

Then the diagonal elements of the transition matrix $P = (p_{ij})$ are determined as follows:

$$p_{ii} = 1 - \sum_{j \neq i} \varepsilon a_{ij}, \quad i = 1, \ldots, n \tag{18}$$

and, in a matrix form,

$$P = I - \varepsilon L(G),$$

where $L(G)$ is the Laplacian matrix of G.

The maximum value of ε that guarantees correctness, i.e., the nonnegativity of the diagonal entries (18) for all simple graphs on n vertices, is obviously $\varepsilon = (n-1)^{-1}$. On the other hand, $\varepsilon = (n-1)^{-1}$ is the only correct ε that allows the self-transition probabilities p_{ii} to be zero. Therefore it makes sense to consider this value of ε and the Markov chain with transition matrix

$$P = I - (n-1)^{-1} L(G) \tag{19}$$

in more detail.

For this chain, let us examine random walks with a random number of steps. Namely, consider a sequence of independent Bernoulli trials indexed by $0, 1, 2, \ldots$ with a certain success probability q. Suppose that the number of steps in a random walk equals the trial

[2] There are two popular methods of attaching a Markov chain to a graph. The first one is based on (17); for any undirected graph it provides a symmetric transition matrix with, in general, nonzero diagonal. The second method assumes that $p_{ij} = a_{ij} / \sum_{k=1}^{n} a_{ik}$. For an undirected graph without loops it generally provides a nonsymmetric transition matrix with zero diagonal.

number of the first success. Then the number of steps, K, is a geometrically distributed random variable:

$$\Pr\{K=k\}=q(1-q)^k,\quad k=0,1,2,\ldots.$$

Suppose that $q=1/n$. For this value, the expected number of steps is $n-1$, which is the number of edges in every spanning tree of G. Then

$$\Pr\{K=k\}=\frac{1}{n}\left(1-\frac{1}{n}\right)^k,\quad k=0,1,2,\ldots. \tag{20}$$

Let $Q=(q_{ij})$ be the matrix with entries

$$q_{ij}=\Pr\{X_K=j\mid X_0=i\},\quad i,j=1,\ldots,n, \tag{21}$$

where X_k is the state of the Markov chain under consideration at step k, i.e., Q is the transition matrix of the overall random walk with a random number of steps K.

Theorem 4 *For a graph G on n vertices and the corresponding Markov chain whose transition matrix is (19), let Q be the transition matrix (21) of the overall random walk whose number of steps is geometrically distributed with parameter $1/n$. Then $Q=F$, where $F=\frac{(f_{ij})_{n\times n}}{f}$ is the doubly stochastic matrix of G.*

Proof. Since the spectral radius of P is 1, for every q such that $0<s<1$

$$\sum_{k=0}^{\infty}(sP)^k=(I-sP)^{-1}$$

holds. Consequently, using the formula of total probability, (20), (19) and the matrix forest theorem (1) we obtain

$$\begin{aligned} Q &= \sum_{k=0}^{\infty}\Pr\{K=k\}\,P^k = \sum_{k=0}^{\infty}\frac{1}{n}\left(1-\frac{1}{n}\right)^k P^k \\ &= \frac{1}{n}\left(I-\left(1-\frac{1}{n}\right)P\right)^{-1}=(I+L)^{-1}=F. \end{aligned}$$

$\square$

By virtue of Theorem 4, if a “golden extrovert” walks randomly in accordance with the above model, she eventually finds herself on a visit ϕ times more often than at home, whereas for a “golden introvert” the situation is opposite.

5 A concluding note: a communicative interpretation of the doubly stochastic graph matrix

In closing, let us mention an interpretation of the doubly stochastic graph matrix in terms of information dissemination. Suppose that a sequence of information units (or ideas) are transmitted through a graph G. A *plan* of information transmission is a rooted forest $\mathsf{F} \in \mathcal{F}(G)$: every information unit (idea) is initially injected into the roots of F; after that it comes to the other vertices along the edges of F. Suppose that every time a possible plan is chosen at random: the probability of every choice is $|\mathcal{F}(G)|^{-1} = f$. Then $\frac{(f_{ij})_{n \times n}}{f}$ is the probability that an information unit arrives at j *from root* i. As a result, for a "golden introvert" the expected proportion of "her own" (injected straight into her mind) ideas to adopted ideas is ϕ, whereas for a "golden extrovert" the proportion is inverse.

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