CHIP-FIRING GAMES, G-PARKING FUNCTIONS, AND AN EFFICIENT BIJECTIVE PROOF OF THE MATRIX-TREE THEOREM

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Abstract. Kirchhoff's matrix-tree theorem states that the number of spanning trees of a graph G is equal to the value of the determinant of the reduced Laplacian of G. We outline an efficient bijective proof of this theorem, by studying a canonical finite abelian group attached to G whose order is equal to the value of same matrix determinant. More specifically, we show how one can efficiently compute a bijection between the group elements and the spanning trees of the graph. The main ingredient for computing the bijection is an efficient algorithm for finding the unique G-parking function (reduced divisor) in a linear equivalence class defined by a chip-firing game. We also give applications, including a new and completely algebraic algorithm for generating random spanning trees. Other applications include algorithms related to chip-firing games and sandpile group law, as well as certain algorithmic problems about the Riemann-Roch theory on graphs.

1. Introduction

1.1. **Overview.** Every graph G has a canonical finite abelian group attached to it. This group has appeared in the literature under many different names; in theoretical physics it was first introduced as the "abelian sandpile group" or "abelian avalanche group" in the context of self-organized critical phenomena ([4, 20, 21]). In arithmetic geometry, this group appeared as the "group of components" in the study of degenerating algebraic curves ([30]). In algebraic graph theory this group appeared under the name "Jacobian group" or "Picard group" in the study of flows and cuts in graphs ([3]). The study of a certain chip-firing game on graphs led to the definition of this group under the name "critical group" ([9, 10]).

The order of this group is equal to the value of the determinant of the reduced Laplacian of G (see, e.g. Lemma 2.3). We know from Kirchhoff's famous matrix-tree theorem that the value of the same determinant gives the number of spanning trees of the graph ([27]). So, one might wonder whether there is a nice and explicit bijection between the elements of the group and the spanning trees of the graph; existence of such a bijection would independently prove the matrix-tree theorem, and might have other algorithmic consequences. It is rather clear that such a bijection cannot be fully canonical, as that would imply a particular spanning tree is distinguished, and corresponds to the identity of the group. Therefore, one needs to make some choices to be able to write down a bijection. If one fixes a vertex q, then there is a canonical representative for each element of the group, called the *G-parking function* (based at q) or *q-reduced divisor* (see, e.g., [21, 19, 31, 5] , or Proposition 2.5). The main result of this paper is an algorithm to find this canonical representative efficiently. Once we have this, we can use one of the bijections in [12, 18, 16, 7] to find the corresponding spanning tree.

This easy-to-compute bijection from the "Jacobian group" to the set of spanning trees of the graph gives new insight into the matrix-tree theorem. It also may lead to new algorithms for graphs and

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