AROUND MATRIX-TREE THEOREM

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ABSTRACT. Generalizing the classical matrix-tree theorem we provide a formula counting subgraphs of a given graph with a fixed 2-core. We use this generalization to obtain an analog of the matrix-tree theorem for the root system D_n (the classical theorem corresponds to the A_n-case). Several byproducts of the developed technique, such as a new formula for a specialization of the multivariate Tutte polynomial, are of independent interest.

1. Introduction

Let us first fix some definitions and notation to be used throughout the paper. The main object of our study will be an undirected graph G without multiple edges. It is understood as a subset $G \subset \{\{i,j\} \mid i,j \in \{1,2,\dots,n\}\}$, where elements of $\{1,2,\dots,n\}$ are vertices and elements of G itself are edges. Informally speaking, this means that we mark (i.e. distinguish) vertices but not edges of G (except for Section 6 where an edge labeling will be used). Usually we will assume that G contains no loops, i.e. edges $\{i,i\}$. Directed graphs (appearing in Sections 2 and 5 for technical purposes) are subsets of $\{1,2,\dots,n\}^2$. Since a graph is understood as a set of edges, notation $F \subset G$ means that F is a subgraph of G.

We will denote by $n = v(G)$ the number of vertices of G, by $\#G = e(G)$ the number of its edges, and by $k(G)$ the number of connected components. For every connected component $G_i \subset G$ $(i = 1,\dots,k(G))$ it will be useful to consider its Euler characteristics $\chi(G_i) = v(G_i) - e(G_i)$. A connected graph containing no cycles will be called a *tree*, a disconnected one, a *forest*. Note that the absence of cycles is equivalent to the equality $\chi(G_i) = 1$ for all i; if cycles are present then $\chi(G_i) \le 0$.

We will usually supply edges of the graph G with weights. A weight $w_{ij} = w_{ji}$ of the edge $\{i,j\}$ is an element of any algebra $\mathcal{A}$. For a subgraph $F \subset G$ denote $w(F) \stackrel{\text{def}}{=} \prod_{\{i,j\}\in F} w_{ij}$; call it the *weight* of F. For any set U of subgraphs of G call the expression $Z(U) = \sum_{F\in U} w(F)$ the *statistical sum* of U. (By definition, we assume $w_{ij} = 0$ if G contains no edge $\{i,j\}$.)

To a graph G with weighted edges one associates its *Laplacian matrix* L_G. It is a symmetric $(n\times n)$-matrix with the elements

$$(L_G)_{ij} = \begin{cases} -w_{ij}, & i \ne j,\\ \sum_{k\ne i} w_{ik}, & i = j.\end{cases}$$

The Laplacian matrix is degenerate; its kernel always contains the vector $(1,1,\dots,1)$. However, its principal minors are generally nonzero and enter the classical matrix-tree theorem whose first version was proved by G. Kirchhoff in 1847:

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